Making Category Theory Accessible

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Category Theory

Category theory is a highly general way of thinking about mathematics (and many other things)

- Able to prove general theorems that apply to many situations
- Arguably the most popular framework for thinking about programming languages
- Can be quite difficult to grasp
- Highly abstract, many implied definitions and concepts, confusing terminology
- “Hardened against understanding by outsiders”
- “A good way to make simple proofs inscrutable”

Can category theory be made accessible?

Category Theory Sources

Books:

- Awodey, Steve. *Category Theory, Second Edition*. Oxford Logic Guides. (Best, in my opinion)
- Pierce, Benjamin. *Basic Category Theory for Computer Scientists*. MIT Press. (Good guide to understanding *The Category Theoretic Solution to Recursive Domain Equations*)
- MacLane, Saunders. *Categories for the Working Mathematician*. Springer. (Nearly impenetrable)

Papers:

- Smyth, Michael and Plotkin, Gordon. *The Category-Theoretic Solution of Recursive Domain Equations*.
- Baez, John. *Physics, Topology, Logic and Computation: A Rosetta Stone*.

Code: Anything written by Ed Kmett

Making Category Theory Accessible

Hypothesis: Category theory could be taught to high-schoolers

- Not terribly more complex than geometry
- Based on fairly intuitive notions (graphs, paths, etc.)
- Make "missing" elements of theory explicit and recognized
- Suggest alternate terminology
- Foundation: a framework for generalizing proofs
- Teach like software engineering: start with concrete examples, *then* abstract them

Making Category Theory Accessible

This presentation breaks from "classical" presentation of category theory. It aims to give a reformulated presentation, and hopefully teach people about category theory in the process.

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Categories

A category can viewed as a (directed) graph with an equivalence relation on paths

- "Objects" in category theory are nodes in a graph
- Graphs have edges between nodes, sequences of edges are paths
- Categories have an *equivalence relation* on paths
- Every object has an id edge, equivalence relation ignores id
- "Arrows" in category theory are *sets of equivalent paths*
- Phrase "there exists a unique arrow from a to b" actually means "all paths from a to b are equivalent"

Reachability Graphs

- The *reachability graph* an edge from N_1 to N_2 if a *path* exists between N_1 and N_2 in the base graph.
- If we have complex edge datatypes, we can compute a *flow graph* if the edge data supports a certain set of operations (*ω-continuous semiring*)
- We can form a *path graph*, where there is an edge from each N_1 to N_2 labeled with a regular expression showing all paths from N_1 to N_2
- We can factor the path sets into equivalence classes based on an equivalence relation; this gives us a representation of a category

Using Category Theory

Category theory can be seen as a method for proving things about things by proving things about graphs.

- Prove things about categories (graphs) using only categorical properites
- Show how to characterize a given structure as a category (graph)
- You then get proofs of all the categorical properties

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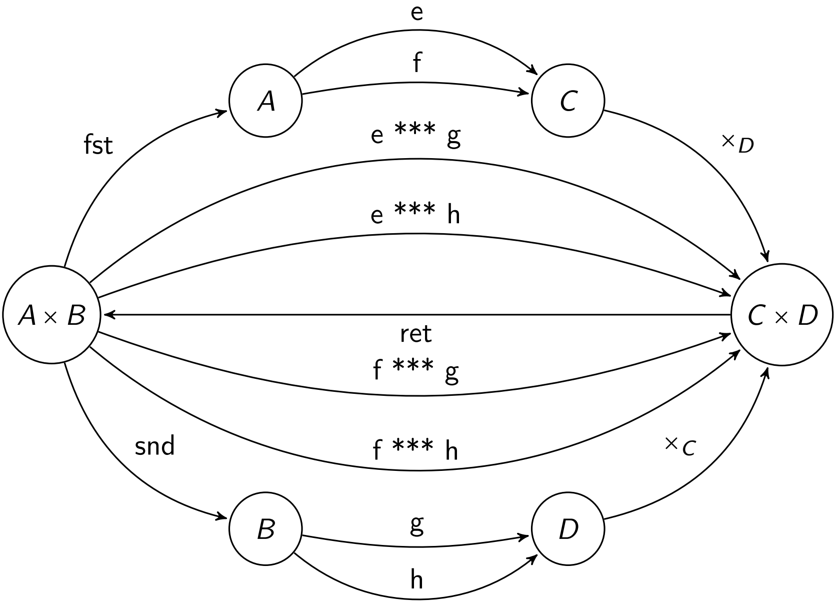
Example of Characterization as a Graph

We'll characterize a simple Haskell API as a graph

```
(***) :: (w -> x) -> (y -> z) -> (w, x) -> (y, z)

e :: A -> C

f :: A -> C

g :: B -> D

h :: B -> D

ret :: (C, D) -> (A, B)
```

Graph



Diagrams

Category theory often uses *diagrams*.

- Diagrams are a subgraph, often representing a particular property
- "This diagram commutes" means "all paths between any N_1 and N_2 are equivalent"
- In other words, the path sets in the path-graph fall into the same equivalence class

Product Diagram

All paths between any two nodes in this diagram are equivalent:

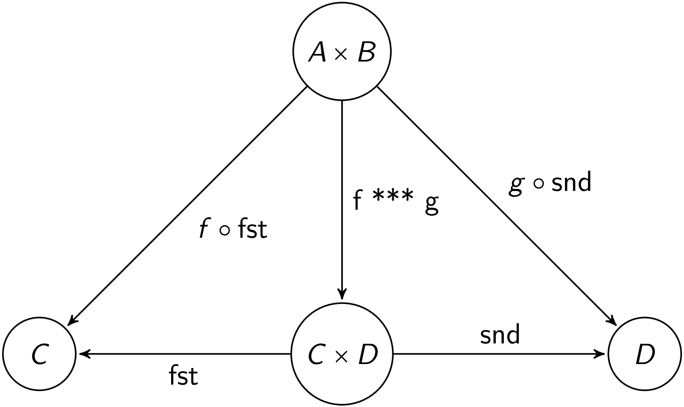


Diagram Chasing

"Diagram chase" is a proof technique common throughout category theory

- Define it clearly by proving that every path between any two nodes is equivalent ("the diagram commutes")
- Concrete semantics: prove equivalence of the enumeration of paths between
- Most general decision procedure: solve for the path-graph (gives us a regex for all paths), then prove equivalence of all paths produced by the regex for each pair of nodes
- Simpler decision procedures exist for subclasses of graphs (i.e. partial orders)

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Inferences About Graphs (Categories)

- Formal reasoning makes use of rules of the form $A \to B$ (A implies B)
- We've talked about characterizing things as graphs (basic categories)
- We've talked about representing path equivalence properties as graphs ("this diagram commutes")
- We need a construct of the form "this property about a graph implies this other property about a graph"

Implications over Graphs

Implications on graphs have three key components:

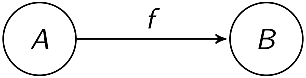
- **Assumptions**: "If we have this structure..."
- **Quantification**: "then for all situations that look like this..."
- **Conclusions**: "this property holds"

Example: Universal Properties

Universal properties can be decomposed into this form, at which point their meaning becomes much clearer (or so I think)

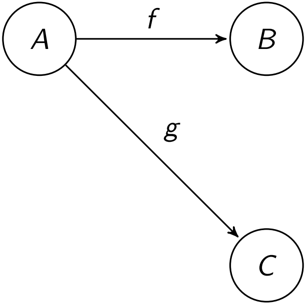
Universal Property (Assumption)

Suppose there is a path f from A to B...



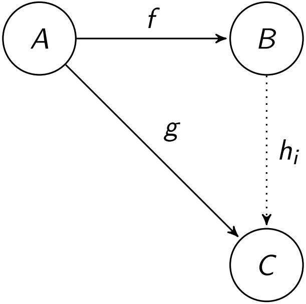
Universal Property (Quantification)

Then for C such that there is a path g from A to C...



Universal Property (Conclusions)

All paths h_i from B to C are equivalent (and at least one h_i exists), and $h_i \circ f = g$ (diagram commutes):



Pedagogical Caveats

- Notice that I've omitted the functor aspect of universal properties
- The implication-like structure is a key point that existing category theory texts fail to properly explain (in my opinion)
- Adding functors into the presentation only confuses things
- Suggests that the "structural" aspect of universal properties is separable from the "functorial" aspect
- Can we separate the "structural" aspects from the "functorial" aspects for other constructions/properties?

Pedagogical Caveats

- We could go on a *tour de force* of the common constructions/properties (products/coproducts, equalizers/coequalizers, limits/colimits) this way
- Awodey gets the order of presentation for these basically right
- Propose renaming "coequalizers" to "normalizers"
- Limits/colimits are backwards
- Propose "categorical limits/mathematical inverse limits", "categorical colimits/mathematical limits"
- Transition into thinking about functors/natural transforms as categories needs to be handled very carefully

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Functors

- A functor is a mapping from one graph to another that preserves all nodes and paths in the source graph
- Functors can be introduced in the context of "representations" of one object in another
- This notion of representation is an *extremely* important idea in mathematics
- Hypothesis: graphs provide an excellent way to demonstrate the idea of representations, *particularly* if we're also teaching students how to prove things by representing problems as a graph

On Functors and (Mathematical) Limits

- Functors as representations opens the door to an effective presentation of infinite objects
- Limits of sequences of graphs, where each element of the sequence has a representation of each previous element
- The natural numbers are a great starting example for this, can go from there to more complex examples
- Simple framework for introducing very powerful ideas (denotational semantics, infinitely-generated algebras, etc.)

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Conclusion

- This represents ongoing work to take a difficult subject and make it accessible (nowhere near complete!)
- Category theory is actually not all that complicated, just needs a major overhaul of presentation
- Graphs are intuitive, can be connected to "natural" human cognitive abilities
- Could potentially be a powerful tool for introducing beginning higher math students to core ideas